Certain Family Of Univalent Functions Associated With Subordination

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Abstract: The objective of this chapter is to introduce a new class $H(\alpha,,\mu,\lambda,A,B)$ of univalent and analytic function p(z)=z+$\sum_{n=1}^{\infty} a_n z^n$ and obtained the results of coefficient estimates, growth and distortion theorem, radii of close to convexity, starlikeness and convexity, clousure theorem, weighted mean ,arithmetic mean, linear combination for the class, applications of fractional calculus and its certain properties for the class.

Keywords: Analytic functions, univalent functions, starlike function,convex function, subordination, closed to convex function, Application of fractional calculus.

INTRODUCTION:

Let A denote the class of function of the form

p(z)= z+$\sum_{n=2}^{\infty} a_n z^n$, (1)

which are univalent in the unit disc

$U = \{z: z \in C \ and \ |z| < 1\}$.

Definition (i):A function p$\in A$ is said to be starlike function of order γ,such that $0 \leq \gamma < 1$ if

$$\mathrm{Re}\left(\frac{zP'(z)}{P(z)}\right) > \gamma \text{, for all } z \in U$$

Definition (ii): A function p$\in A$ is said to be convex function of order γ , *such that* $0 \leq \gamma < 1$ if

$$\mathrm{Re}\left(1 + \frac{zP'(z)}{P(z)}\right) > \gamma \text{, } z \in U$$

Definition (iii): A function p$\in A$ is said to be close to convex function of order γ, *such that* $0 \leq \gamma < 1$ if

$$\mathbf{Re}\left(\frac{p'(z)}{z^{p-1}}\right) > \gamma \text{, for all } z \in U$$

Definition (iv): (subordinate principle) For two functions p and q analytic in U , we say that the function p(z) is subordinate to q(z) in U and write

$p \prec q$ or p(z) $\prec q$(z) (z $\in$ U)

with $\omega(0) = 0 \ and \ |\omega(z)| < 1 \ (z \in U)$ Such that

f(z) = g($\omega(z)$) (z$\in U$)

In particular, if the function q is univalent in U the above subordination is equivalent to

p(0) = q(0) and p(U)$\subset q(U)$

Definition (v) : A function p(z)$\in A$ is in the class $H(\alpha,,\mu,\lambda,A,B)$ which satisfies

$$\left|\frac{z\frac{p'(z)}{p(z)}-1}{(1-\alpha)\mu+\alpha\lambda-\lambda z\frac{p'(z)}{p(z)}}\right| \prec \frac{1+Az}{1+Bz} \quad \text{For } 0\leq \alpha < 1,\ 0\leq \lambda < \mu \leq 1,$$

$-1\leq B < A \leq 1$ (1.2)

2.**Basic Properties For The Class $H(\alpha,,\mu,\lambda,A,B)$**

Theorem (I):A function p(z) = z+$\sum_{n=1}^{\infty} a_n z^n$ is in $H(\alpha,,\mu,\lambda,A,B)$ if and only if

$$\sum_{n=2}^{\infty}\left[(n-1)-\left((1-\alpha)\mu+\alpha\lambda+\lambda n\right)+(n+1)B\right]a_n$$

$$\leq (A+1)\left((1-\alpha)\mu+\alpha\lambda-\lambda\right) \quad (1.3)$$

Proof: Suppose p(z)$\in H(\alpha,,\mu,\lambda,A,B)$

Therefore from equation (1.2) we have

$$f(z) = \left|\frac{z\frac{p'(z)}{p(z)}-1}{(1-\alpha)\mu+\alpha\lambda-\lambda z\frac{p'(z)}{p(z)}}\right| \prec \frac{1+Az}{1+Bz}$$

$$f(z) = \frac{1+Aw(z)}{1+Bw(z)}$$

$$f(z)[1+Bw(z)] = 1+Aw(z)$$

$$w(z)[Bf(z)-A] = 1-f(z)$$

$$w(z) = \frac{f(z)-1}{A-Bf(z)}$$

$$|w(z)| < 1$$

$$\left|\frac{f(z)-1}{A-Bf(z)}\right| < 1$$

$$\therefore \left|\frac{\left[\frac{z\frac{p'(z)}{p(z)}-1}{(1-\alpha)\mu+\alpha\lambda-\lambda z\frac{p'(z)}{p(z)}}\right]-1}{A-B\left[\frac{z\frac{p'(z)}{p(z)}-1}{(1-\alpha)\mu+\alpha\lambda-\lambda z\frac{p'(z)}{p(z)}}\right]}\right| < 1$$

$$\therefore \left|\frac{zp'(z)-p(z)-p(z)\{((1-\alpha)\mu+\alpha\lambda\}+\lambda zp'(z)}{Ap(z)\{((1-\alpha)\mu+\alpha\lambda\}-A\lambda p'(z)-Bzp'(z)+Bp(z)}\right| < 1$$

Now $p(z) = z+\sum_{n=2}^{\infty} a_n z^n$

$\therefore\ zp'(z) = z+\sum_{n=2}^{\infty} a_n n z^n$

$\therefore zp'(z)-p(z)-p(z)\{((1-\alpha)\mu+\alpha\lambda\}+\lambda zp'(z)$

$= z+\sum_{n=2}^{\infty} a_n n z^n - z-\sum_{n=2}^{\infty} a_n z^n$-

$(z+\sum_{n=2}^{\infty} a_n z^n)\{((1-\alpha)\mu+\alpha\lambda\}+\lambda z+\sum_{n=2}^{\infty} a_n \lambda n z^n$

$= \sum_{n=2}^{\infty}\left[n-1-\left((1-\alpha)\mu+\alpha\lambda\right)+\lambda n\right]a_n z^n$-

$\left[\left((1-\alpha)\mu+\alpha\lambda\right)-\lambda\right]z$

And

$Ap(z)\{((1-\alpha)\mu+\alpha\lambda\}-A\lambda zp'(z)-Bzp'(z)+Bp(z)$

$=A(z+\sum_{n=2}^{\infty} a_n z^n)\{((1-\alpha)\mu+\alpha\lambda\}-A\lambda(z+\sum_{n=2}^{\infty} a_n n z^n)$-

$B(z+\sum_{n=2}^{\infty} a_n n z^n)+B(z+\sum_{n=2}^{\infty} a_n z^n)$

$=\sum_{n=2}^{\infty}\left[A\left((1-\alpha)\mu+\alpha\lambda-\lambda n\right)-B(n+1)\right]a_n z^n$

$+A[((1-\alpha)\mu+\alpha\lambda-\lambda]z$

$$\therefore \left|\frac{\sum_{n=2}^{\infty}\left[n-1-\left((1-\alpha)\mu+\alpha\lambda\right)+\lambda n\right]a_n z^n-\left[\left((1-\alpha)\mu+\alpha\lambda\right)-\lambda\right]z}{\sum_{n=2}^{\infty}\left[A\left((1-\alpha)\mu+\alpha\lambda-\lambda n\right)-B(n+1)\right]a_n z^n+A[((1-\alpha)\mu+\alpha\lambda-\lambda]z}\right| < 1$$

Since Re(z)<$|z|$.After choosing the values of z on real axis and letting z→ 1 we get

$\sum_{n=2}^{\infty}\left[n-1-\left((1-\alpha)\mu+\alpha\lambda\right)+\lambda n\right]a_n-\left[\left((1-\alpha)\mu+\alpha\lambda\right)-\lambda\right]$

$\leq \sum_{n=2}^{\infty}\left[A\left((1-\alpha)\mu+\alpha\lambda-\lambda n\right)-B(n+1)\right]a_n$

$+A[((1-\alpha)\mu+\alpha\lambda-\lambda]$

$\therefore \sum_{n=2}^{\infty}\left[n-1-\left((1-\alpha)\mu+\alpha\lambda\right)+\lambda n\right]a_n -$

$\sum_{n=2}^{\infty}\left[A\left((1-\alpha)\mu+\alpha\lambda-\lambda n\right)-B(n+1)\right]a_n$

$\leq A[((1-\alpha)\mu+\alpha\lambda-\lambda]+\left[\left((1-\alpha)\mu+\alpha\lambda\right)-\lambda\right]$

$\sum_{n=2}^{\infty}\left[(n-1)-\left((1-\alpha)\mu+\alpha\lambda+\lambda n\right)(A+1)+(n+1)B\right]a_n$

$\leq (A+1)\left((1-\alpha)\mu+\alpha\lambda-\lambda\right)$

Corollary(I): If p(z)$\in H(\alpha,,\mu,\lambda,A,B)$ then

$$a_n \leq \frac{(A+1)\left((1-\alpha)\mu+\alpha\lambda-\lambda\right)}{[(n-1)-\left((1-\alpha)\mu+\alpha\lambda+\lambda n\right)(A+1)+(n+1)B]} \quad (1.4)$$

And the equality holds for

$$p(z) = z+\frac{(A+1)\left((1-\alpha)\mu+\alpha\lambda-\lambda\right)}{[(n-1)-\left((1-\alpha)\mu+\alpha\lambda+\lambda n\right)(A+1)+(n+1)B]}z^n \quad (1.5)$$

Theorem(II): If p(z)$\in H(\alpha,,\mu,\lambda,A,B)$ then

$$|z|+|z^2|\frac{(A+1)\left((1-\alpha)\mu+\alpha\lambda-\lambda\right)}{[1-\left((1-\alpha)\mu+\alpha\lambda+2\lambda\right)(A+1)+3B]} \leq |p(z)|$$

$$\leq |z|-|z^2|\frac{(A+1)\left((1-\alpha)\mu+\alpha\lambda-\lambda\right)}{[1-\left((1-\alpha)\mu+\alpha\lambda+2\lambda\right)(A+1)+3B]} \quad (1.6)$$

Proof: p(z)$\in H(\alpha,,\mu,\lambda,A,B)$ Therefore from theorem(I)

$$\sum_{n=2}^{\infty} a_n \leq \frac{(A+1)\left((1-\alpha)\mu+\alpha\lambda-\lambda\right)}{[(n-1)-\left((1-\alpha)\mu+\alpha\lambda+\lambda n\right)(A+1)+(n+1)B]}$$

$$p(z) = z+\sum_{n=2}^{\infty} a_n z^n$$

$$|p(z)| \geq |z|+|z^2|\sum_{n=2}^{\infty}|a_n|$$

$$|p(z)| \geq |z|+|z^2|\frac{(A+1)\left((1-\alpha)\mu+\alpha\lambda-\lambda\right)}{[1-\left((1-\alpha)\mu+\alpha\lambda+2\lambda\right)(A+1)+3B]}$$

And

$$|p(z)| \leq |z|-|z^2|\frac{(A+1)\left((1-\alpha)\mu+\alpha\lambda-\lambda\right)}{[1-\left((1-\alpha)\mu+\alpha\lambda+2\lambda\right)(A+1)+3B]}$$

Therefore

$$|z|+|z^2|\frac{(A+1)\left((1-\alpha)\mu+\alpha\lambda-\lambda\right)}{[1-\left((1-\alpha)\mu+\alpha\lambda+2\lambda\right)(A+1)+3B]} \leq |p(z)|$$

$$\leq |z|-|z^2|\frac{(A+1)\left((1-\alpha)\mu+\alpha\lambda-\lambda\right)}{[1-\left((1-\alpha)\mu+\alpha\lambda+2\lambda\right)(A+1)+3B]}$$

Theorem (III): If p(z)$\in H(\alpha,,\mu,\lambda,A,B)$ then

$$1+|x|\frac{2(A+1)\left((1-\alpha)\mu+\alpha\lambda-\lambda\right)}{[1-\left((1-\alpha)\mu+\alpha\lambda+2\lambda\right)(A+1)+3B]} \leq |p'(x)|$$

$$\leq 1-|x|\frac{2(A+1)\left((1-\alpha)\mu+\alpha\lambda-\lambda\right)}{[1-\left((1-\alpha)\mu+\alpha\lambda+2\lambda\right)(A+1)+3B]} \quad (1.7)$$

Proof: p(z)$\in H(\alpha,,\mu,\lambda,A,B)$ Therefore from theorem (I)

$$\sum_{n=2}^{\infty} a_n \leq \frac{(A+1)\left((1-\alpha)\mu+\alpha\lambda-\lambda\right)}{[(n-1)-\left((1-\alpha)\mu+\alpha\lambda+\lambda n\right)(A+1)+(n+1)B]}$$

$$p(z) = z+\sum_{n=2}^{\infty} a_n z^n$$

$$p'(z) = 1+\sum_{n=2}^{\infty} a_n n z^{n-1}$$

$$|p'(z)| \geq 1 + 2|z| \sum_{n=2}^{\infty} |a_n| \geq 1 + |z| \frac{2(A+1)((1-\alpha)\mu+\alpha\lambda-\lambda)}{[1-((1-\alpha)\mu+\alpha\lambda+2\lambda)(A+1)+3B]}$$

And

$$|p'(z)| \leq 1 - 2|z| \sum_{n=2}^{\infty} |a_n| \leq 1 - |z| \frac{2(A+1)((1-\alpha)\mu+\alpha\lambda-\lambda)}{[1-((1-\alpha)\mu+\alpha\lambda+2\lambda)(A+1)+3B]}$$

Therefore

$$1 + |z| \frac{2(A+1)((1-\alpha)\mu+\alpha\lambda-\lambda)}{[1-((1-\alpha)\mu+\alpha\lambda+2\lambda)(A+1)+3B]} \leq |p'(z)|$$

$$\leq 1 - |z| \frac{2(A+1)((1-\alpha)\mu+\alpha\lambda-\lambda)}{[1-((1-\alpha)\mu+\alpha\lambda+2\lambda)(A+1)+3B]}$$

Theorem (IV): Suppose the function p(z) is defined by (i) is in the class then p(z) is starlike of order γ, such that $0 \leq \gamma < 1$ in $|x| < R_1$ where

$$R_1 = \inf_n \left\{ \left(\frac{1-\gamma}{n-2+\gamma}\right) \left(\frac{[(n-1)-((1-\alpha)\mu+\alpha\lambda+\lambda n)(A+1)+(n+1)B]}{(A+1)((1-\alpha)\mu+\alpha\lambda-\lambda)}\right) \right\}^{\frac{1}{n-1}} \quad (1.8)$$

Proof: We must show that

$$\left| \frac{zp'(z)}{p(z)} - 1 \right| < 1-\gamma \quad (1.9)$$

$$\therefore \left| \frac{\sum_{n=2}^{\infty}(n-1)a_n z^n}{z+\sum_{n=2}^{\infty} a_n z^n} \right| < 1-\gamma$$

$$\sum_{n=2}^{\infty}(n-1)|a_n||z|^n \leq (1-\gamma)(|z| + \sum_{n=2}^{\infty}|a_n||z|^n)$$

$$\sum_{n=2}^{\infty}(n-1)|a_n||z|^n - (1-\gamma)(\sum_{n=2}^{\infty}|a_n||z|^n) \leq (1-\gamma)\,|z|$$

$$\sum_{n=2}^{\infty} \frac{(n-2+\gamma)}{(1-\gamma)} |a_n||z|^{n-1} \leq 1 \quad (2.1)$$

From Theorem (I) we have

$$\sum_{n=2}^{\infty} \frac{[(n-1)-((1-\alpha)\mu+\alpha\lambda+\lambda n)(A+1)+(n+1)B]}{(A+1)((1-\alpha)\mu+\alpha\lambda-\lambda)} a_n \leq 1 \quad (2.2)$$

Hence by using (2.1) and (2.2) we get

$$\frac{(n-2+\gamma)}{(1-\gamma)} |z|^{n-1} \leq \frac{[(n-1)-((1-\alpha)\mu+\alpha\lambda+\lambda n)(A+1)+(n+1)B]}{(A+1)((1-\alpha)\mu+\alpha\lambda-\lambda)}$$

$$|z|^{n-1} \leq \left(\frac{1-\gamma}{n-2+\gamma}\right) \left(\frac{[(n-1)-((1-\alpha)\mu+\alpha\lambda+\lambda n)(A+1)+(n+1)B]}{(A+1)((1-\alpha)\mu+\alpha\lambda-\lambda)}\right)$$

$$|z| \leq \left(\left(\frac{1-\gamma}{n-2+\gamma}\right) \left(\frac{[(n-1)-((1-\alpha)\mu+\alpha\lambda+\lambda n)(A+1)+(n+1)B]}{(A+1)((1-\alpha)\mu+\alpha\lambda-\lambda)}\right) \right)^{\frac{1}{n-1}}$$

This completes the proof.

Now, Since p is convex iff zp' is starlike, then we have

Theorem(V): Suppose the function p(z) is defined by (i) is in the class then p(z) is convex of order γ, such that $0 \leq \gamma < 1$ in $|z| < R_2$ where

$$R_2 = \inf_n \left\{ \left(\frac{1-\gamma}{n(n-2+\gamma)}\right) \left(\frac{[(n-1)-((1-\alpha)\mu+\alpha\lambda+\lambda n)(A+1)+(n+1)B]}{(A+1)((1-\alpha)\mu+\alpha\lambda-\lambda)}\right) \right\}^{\frac{1}{n-1}}$$

Proof: We must show that

$$\left| \frac{zp''(z)}{p'(z)} \right| < 1-\gamma \quad (2.4)$$

$$\therefore \left| \frac{\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{1+\sum_{n=2}^{\infty} n a_n z^{n-1}} \right| < 1-\gamma$$

$$\sum_{n=2}^{\infty} n(n-1)|a_n||z|^{n-1} \leq (1-\gamma)(1+\sum_{n=2}^{\infty} n\,|a_n||z|^{n-1})$$

$$\sum_{n=2}^{\infty} n(n-1)|a_n||z|^{n-1} - (1-\gamma)(\sum_{n=2}^{\infty} n\,|a_n||z|^{n-1}) \leq (1-\gamma)$$

$$\sum_{n=2}^{\infty} \frac{n(n-2+\gamma)}{(1-\gamma)} |a_n||z|^{n-1} \quad (2.5)$$

From Theorem (I) we have

$$\sum_{n=2}^{\infty} \frac{[(n-1)-((1-\alpha)\mu+\alpha\lambda+\lambda n)(A+1)+(n+1)B]}{(A+1)((1-\alpha)\mu+\alpha\lambda-\lambda)} a_n \leq 1 \quad (2.6)$$

Hence by using (2.5) and (2.6) we get

$$\frac{n(n-2+\gamma)}{(1-\gamma)} |z|^{n-1} \leq \frac{[(n-1)-((1-\alpha)\mu+\alpha\lambda+\lambda n)(A+1)+(n+1)B]}{(A+1)((1-\alpha)\mu+\alpha\lambda-\lambda)}$$

$$|z|^{n-1} \leq \left(\frac{1-\gamma}{n(n-2+\gamma)}\right) \left(\frac{[(n-1)-((1-\alpha)\mu+\alpha\lambda+\lambda n)(A+1)+(n+1)B]}{(A+1)((1-\alpha)\mu+\alpha\lambda-\lambda)}\right)$$

$$|z| \leq \left(\left(\frac{1-\gamma}{n(n-2+\gamma)}\right) \left(\frac{[(n-1)-((1-\alpha)\mu+\alpha\lambda+\lambda n)(A+1)+(n+1)B]}{(A+1)((1-\alpha)\mu+\alpha\lambda-\lambda)}\right) \right)^{\frac{1}{n-1}}$$

Which complete the proof.

Theorem (VI): Suppose the function p(z) is defined by (i) is in the class then p(z) is close to convex of order γ, such that $0 \leq \gamma < 1$ in $|x| < R_3$ where

$$R_3 = \inf_n \left\{ \left(\frac{(1-\gamma)[(n-1)-((1-\alpha)\mu+\alpha\lambda+\lambda n)(A+1)+(n+1)B]}{n(A+1)((1-\alpha)\mu+\alpha\lambda-\lambda)} \right) \right\}^{\frac{1}{n-1}} \quad (2.7)$$

Proof: It is sufficient to show that $|p'(z) - 1| \le 1 - \gamma$ $(0 \le \gamma < 1)$, $|z| < R_3$

Thus $|p'(z) - 1| - |\sum_{n=2}^{\infty} na_n z^{n-1}| \le \sum_{n=2}^{\infty} na_n |z|^{n-1}$

Thus

$$|p'(z) - 1| \le 1 - \gamma \text{ if } \sum_{n=2}^{\infty} \left(\frac{n}{1-\gamma}\right) a_n |z|^{n-1} \le 1 \quad (2.8)$$

Bue theorem (I) confirms that

$$\sum_{n=2}^{\infty} \frac{[(n-1)-((1-\alpha)\mu+\alpha\lambda+\lambda n)(A+1)+(n+1)B]}{(A+1)((1-\alpha)\mu+\alpha\lambda-\lambda)} a_n \le 1$$

Hence (2.8) will be true if

$$\left(\frac{n}{1-\gamma}\right) |z|^{n-1} \le \frac{[(n-1)-((1-\alpha)\mu+\alpha\lambda+\lambda n)(A+1)+(n+1)B]}{(A+1)((1-\alpha)\mu+\alpha\lambda-\lambda)}$$

Or if $|z| \le \left(\frac{(1-\gamma)[(n-1)-((1-\alpha)\mu+\alpha\lambda+\lambda n)(A+1)+(n+1)B]}{n(A+1)((1-\alpha)\mu+\alpha\lambda-\lambda)} \right)^{\frac{1}{n-1}}$ (n≥ 2)

3.Weighted Mean, Arithmetic Mean and Linear Combination

Resently,W.G.Asthan,H.D.Mustafa and E.K.Mouajeeb [2] proved Weighted Mean, Arithmetic Mean and Linear Combination of regular function.

Definition(vi): Let p, q∈D(A,B,δ) then the weighted mean w_{pq} of p and q is defined as

$$w_{pq} = \frac{1}{2}[(1-m)p(z) + (1+m)q(z)], \quad 0<m<1 \quad (3.1)$$

Definition(vii): Let $f_i(z)$= $z^p - \sum_{k=1+p}^{\infty} a_{i,k} z^k$, i = 1,2,3,-----,m be the function in the class D(A,B,δ)

Then the arithmetic mean of f_i (i = 1,2,3…..,m is defined by $g(z)=\frac{1}{m}\sum_{i=1}^{m} f_i(z)$ (3.2)

Theorem (VII): Let p, q∈ $H(\alpha,,\mu,\lambda,A,B)$ .Then the

weighted mean w_{pq} of p and q is also in the class $H(\alpha,,\mu,\lambda,A,B)$.

Proof: By definition 3.1 we have

$$w_{pq} = \frac{1}{2}[(1-m)p(z) + (1+m)q(z)], \quad 0<m<1$$

$$= \frac{1}{2}[(1-m)(z + \sum_{n=2}^{\infty} a_n z^n) + (1+m)(z + \sum_{n=2}^{\infty} b_n z^n)]$$

$$= z + \sum_{n=2}^{\infty} \frac{1}{2}[(1-m)a_n + (1+m)b_n] z^n$$

Since, q∈ $H(\alpha,,\mu,\lambda,A,B)$ so by theorem I we have

$$\sum_{n=2}^{\infty}[(n-1) - ((1-\alpha)\mu + \alpha\lambda + \lambda n)(A+1) + (n+1)B]a_n \le (A+1)((1-\alpha)\mu + \alpha\lambda - \lambda)$$

And

$$\sum_{n=2}^{\infty}[(n-1) - ((1-\alpha)\mu + \alpha\lambda + \lambda n)(A+1) + (n+1)B]b_n \le (A+1)((1-\alpha)\mu + \alpha\lambda - \lambda)$$

Therefore

$$\sum_{n=2}^{\infty}[(n-1) - ((1-\alpha)\mu + \alpha\lambda + \lambda n)(A+1) + (n+1)B]\left[\frac{1}{2}[(1-m)a_n + (1+m)b_n]\right]$$

$$= \frac{1}{2}(1-m)\sum_{n=2}^{\infty}[(n-1) - ((1-\alpha)\mu + \alpha\lambda + \lambda n)(A+1) + (n+1)B]a_n + \frac{1}{2}(1+m)\sum_{n=2}^{\infty}[(n-1) - ((1-\alpha)\mu + \alpha\lambda + \lambda n)(A+1) + (n+1)B]b_n$$

$$\le \frac{1}{2}(1-m)(A+1)((1-\alpha)\mu + \alpha\lambda - \lambda) + \frac{1}{2}(1+m)(A+1)((1-\alpha)\mu + \alpha\lambda - \lambda)$$

$$= (A+1)((1-\alpha)\mu + \alpha\lambda - \lambda)$$

Therefore $w_{pq} \in H(\alpha,,\mu,\lambda,A,B)$

Hence the proof of theorem is completed.

Theorem(VIII): Let $p_i(z) = z + \sum_{n=2}^{\infty} a_{i,n} z^n$, i = 1,2,3,------,m be the functions in the class $H(\alpha,,\mu,\lambda,A,B)$

Then the arithmetic mean of p_i (i=1,2,3,-------m) is defined by $g(z)=\frac{1}{m}\sum_{i=1}^{m} p_i(z)$ is also in the class $H(\alpha,,\mu,\lambda,A,B)$.

Proof: Since $p_i(z) = z + \sum_{n=2}^{\infty} a_{i,n} z^n$, i = 1,2,3,------,m

Therefore

$$g(z) = \frac{1}{m}\sum_{i=1}^{m} p_i(z) \quad (3.3)$$

$$= \frac{1}{m}\sum_{i=1}^{m}\left(z + \sum_{n=2}^{\infty} a_{i,n} z^n\right)$$

$$= z + \sum_{i=1}^{\infty}\left(\frac{1}{m}\sum_{i=1}^{m} a_{i,n}\right) z^n$$

We have $p_i(z) = z + \sum_{n=2}^{\infty} a_{i,n} z^n$, i = 1,2,3,------,m.So by theorem 2.1 we have

$$\sum_{n=2}^{\infty}[(n-1) - ((1-\alpha)\mu + \alpha\lambda + \lambda n)(A+1) + (n+1)B]\left(\frac{1}{m}\sum_{i=1}^{m} a_{i,n}\right)$$

$$= \frac{1}{m}\sum_{i=1}^{m}\left(\sum_{n=2}^{\infty}[(n-1)-((1-\alpha)\mu+\alpha\lambda+\lambda n)(A+1) + (n+1)B]a_{i,n}\right)$$

$\leq \frac{1}{m}\sum_{i=1}^{m}\left((A+1)((1-\alpha)\mu+\alpha\lambda-\lambda)\right)=(A+1)((1-\alpha)\mu+\alpha\lambda-\lambda)$

Hence the proof of the theorem is completed.

Theorem(IX): Let $p_i(z) = z + \sum_{n=2}^{\infty} a_{i,n}z^n$, i = 1,2,3,------,m be the functions in the class $H(\alpha,,\mu,\lambda,A,B)$

then the linear combination of p_i (i=1,2,3,-------m) is defined by

$G(z)=\sum_{i=1}^{m} n_i p_i(z)$ where $\sum_{i=1}^{m} n_i = 1$ (3.4)

is also in the class $H(\alpha,,\mu,\lambda,A,B)$.

Proof: Let $p_i(z) = z + \sum_{n=2}^{\infty} a_{i,n}z^n$, i = 1,2,3,------,m be the functions in the class $H(\alpha,,\mu,\lambda,A,B)$

So by theorem I we have

$\sum_{n=2}^{\infty}[(n-1)-((1-\alpha)\mu+\alpha\lambda+\lambda n)(A+1) + (n+1)B]a_{i,n} \leq (A+1)$

$G(z)=\sum_{i=1}^{m} n_i p_i(z)$

$G(z) = \sum_{i=1}^{m} n_i\left(z + \sum_{n=2}^{\infty} a_{i,n}z^n\right)$

$G(z) = z+\sum_{n=2}^{\infty}\left(\sum_{n=2}^{\infty} n_i a_{i,n}\right)z^n$

So by theorem (I) we have

$\sum_{n=2}^{\infty}[(n-1)-((1-\alpha)\mu+\alpha\lambda+\lambda n)(A+1) + (n+1)B]\left(\sum_{i=1}^{m} n_i a_{i,n}\right)$

$= \sum_{i=1}^{m} n_i\left(\sum_{n=2}^{\infty}[(n-1)-((1-\alpha)\mu+\alpha\lambda+\lambda n)(A+1) + (n+1)B]a_{i,n}\right)$

$$\leq \sum_{i=1}^{m} n_i\,(A+1)((1-\alpha)\mu+\alpha\lambda-\lambda)$$

$= (A+1)((1-\alpha)\mu+\alpha\lambda-\lambda)$

Hence the proof of the theorem is complited.

4.Application of Fractional Calculus and its certain properties:

Various operators of fractional calculus have been studied in the literature rather extensively.

We recall the following definitions

Definition: The integral operator studied by Bernardi is

$$L_c[p] = \frac{a+c}{z^c}\int_0^z f(x)x^{c-1}\,dx$$

Theorem (Z): If p∈ $H(\alpha,,\mu,\lambda,A,B)$ then $L_c[p]$ is also in the class $H(\alpha,,\mu,\lambda,A,B)$

Proof: Let p(z) = z+$\sum_{n=2}^{\infty} a_n z^n$ then

$L_c[p] = \frac{1+c}{z^c}\int_0^z (x + \sum_{n=2}^{\infty} a_n x^n)x^{c-1}\,dx$

$$= \frac{1+c}{z^c}\left[\left(\frac{1}{1+c}x^{1+c} + \sum_{n=2}^{\infty}\frac{1}{n+c}a_n x^{n+c}\right)_0^z\right] \quad (4.1)$$

$= z + \sum_{n=2}^{\infty}\frac{1+c}{n+c}a_n z^n$

Since c>-1, n≥ 2 then $\frac{1+c}{n+c} \leq 1$ so we have

$$\sum_{n=2}^{\infty}\frac{[(n-1)-((1-\alpha)\mu+\alpha\lambda+\lambda n)(A+1)+(n+1)B]}{(A+1)((1-\alpha)\mu+\alpha\lambda-\lambda)}\left(\frac{1+c}{n+c}\right)a_n$$

$$\leq \sum_{n=2}^{\infty}\frac{[(n-1)-((1-\alpha)\mu+\alpha\lambda+\lambda n)(A+1)+(n+1)B]}{(A+1)((1-\alpha)\mu+\alpha\lambda-\lambda)}a_n < 1$$

Therefore $L_c[p] \in H(\alpha,,\mu,\lambda,A,B)$

Theorem (XI): Let p∈ $H(\alpha,,\mu,\lambda,A,B)$ then for every z≥ 0 then the function

$L_x[z]$= (1-x)p(z)+x$\int_0^z \frac{f(y)}{y}$dy is also in the class $H(\alpha,,\mu,\lambda,A,B)$.

Proof: $L_x[z]$= (1-x)(z + $\sum_{n=2}^{\infty} a_n z^n$) +x$\int_0^z \frac{y+\sum_{n=2}^{\infty} a_n y^n}{y}$dy

$= z - xz + \sum_{n=2}^{\infty} a_n z^n - \sum_{n=2}^{\infty} x a_n z^n + xz + \sum_{n=2}^{\infty} a_n \frac{x}{n} z^n$

$$= z + \sum_{n=2}^{\infty}\left(1 - x + \frac{x}{n}\right)a_n z^n \quad (4.2)$$

Since $\left(1 - x + \frac{x}{n}\right) < 1$.Therefore we have

$$\sum_{n=2}^{\infty}\frac{[(n-1)-((1-\alpha)\mu+\alpha\lambda+\lambda n)(A+1)+(n+1)B]\left(1-x+\frac{x}{n}\right)}{(A+1)((1-\alpha)\mu+\alpha\lambda-\lambda)}a_n \leq 1$$

By Theorem (i) we have $L_x[z] \in H(\alpha,,\mu,\lambda,A,B)$.

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